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AN ALGEBRAIC CHARACTERIZATION OF FIXED MODES IN A DECENTRALIZED CONTROL (A SIMPLIFIED EXISTENCE PROOF)

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Abstract—Algebraic characterizations for the existence of *fixed modes* in a linear closed-loop system with linear decentralized feedback controls are now well known. A simplified approach to the existence proof is presented here

Wang and Davision [1] introduced the concept of "fixed modes" to characterize certain natural frequencies arising in linear, time-invariant, finite dimensional plants using linear, finite dimensional decentralized feedback controllers, which are independent of the particular controller used. Anderson and Clements [2] presented a rather elegant set of algebraic tests which provide necessary and sufficient conditions for the existence of closed-loop fixed modes in decentralized control. In this paper, we present a simplified proof of the "basic theorem" from the Anderson-Clements paper, which we hope may provide a clearer intuitive grasp of the nature of fixed modes in decentralized control.

We consider the system

$$\dot{\mathbf{x}} = \mathbf{F}\,\mathbf{x} + \mathbf{G}\,\mathbf{u}, \tag{1}$$

where $\mathbf{F} \in \mathbb{R}^{n\times n}$, $\mathbf{G} \in \mathbb{R}^{n\times m}$. A partitioning of the vector $\mathbf{x} = (\mathbf{x}_1^{\mathsf{T}}, \mathbf{x}_2^{\mathsf{T}}, \dots, \mathbf{x}_p^{\mathsf{T}})$, where $\mathbf{x}_i \in \mathbb{R}^{n_i}$, $n = \sum_{i=1}^{p} n_i$, serves to define the decentralized structure. Let $\mathcal{K}$ be the set of block diagonal matrices

$$\mathcal{K} = \left\{\mathbf{K} \mid \mathbf{K} = \operatorname{diag}(\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_p), \quad \mathbf{K}_i \in \mathbb{R}^{m_i \times n_i}\right\}, \tag{2}$$

where $m = \sum_{i=1}^{p} m_i$. Then the set of *fixed modes* of $\{\mathbf{F}, \mathbf{G}\}$ with respect of $\mathcal{K}$ is defined as

$$\bigwedge(\mathbf{F}, \mathbf{G}, \mathcal{K}) = \bigcap_{\mathbf{K}\in\mathcal{K}} \sigma(\mathbf{F} + \mathbf{G}\,\mathbf{K}), \tag{3}$$

where $\sigma(\cdot)$ denotes the set of eigenvalues of $(\cdot)$.

Let us employ the notation

$$\mathbf{G} = \left[\mathbf{G}_1 \,\vdots\, \mathbf{G}_2 \,\vdots\, \cdots \,\vdots\, \mathbf{G}_p\right], \tag{4}$$

$$s\,\mathbf{I} - \mathbf{F} = \left[\mathbf{H}_1(s) \,\vdots\, \mathbf{H}_2 \,\vdots\, \cdots \,\vdots\, \mathbf{H}_p(s)\right], \tag{5}$$

to state the basic theorem.

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THEOREM 1. *A necessary and sufficient condition for the existence of fixed modes in* $\{\mathbf{F},\mathbf{G}\}$ *with respect to* $\mathcal{K}$ *is that there exists a non-empty subset* $I=\{i_1,i_2,\dots,i_j\}$ *of* $\{1,2,\dots,m\}$ *such that*

$$\mathit{rank}\ \left[\mathbf{H}_{i_1}(\lambda_0) \vdots \mathbf{H}_{i_2}(\lambda_0) \vdots \cdots \vdots \mathbf{H}_{i_j}(\lambda_0) \vdots \mathbf{G}_{i_j} \vdots \cdots \vdots \mathbf{G}_{i_j}\right] < \sum_{i=1}^{j} n_i \tag{6}$$

for some $\lambda_0 \in \sigma(\mathbf{H})$.

PROOF. The sufficient part is rather obvious. By post multiplication of the matrix on the left-hand side of (6) by the matrix $\begin{bmatrix}\mathbf{I}\\ \mathbf{K}_I\end{bmatrix}$ where I is the identity matrix and

$$\mathbf{K}_I = \operatorname{diag}\left(\mathbf{K}_{i_1},\dots,\mathbf{K}_{i_j}\right), \quad \mathbf{K}_{i_k} \in \mathbb{R}^{m_{i_k}\times n_{i_k}},$$

one obtains

$$\text{rank}\ \left[\mathbf{H}_{i_1} + \mathbf{G}_{i_1}\mathbf{K}_{i_1} \vdots \cdots \vdots \mathbf{H}_{i_j} + \mathbf{G}_{i_j}\mathbf{K}_{i_j}\right] < \sum_{k=1}^{j} n_{i_k}, \tag{7}$$

which implies $(\lambda_0\,\mathbf{I} - \mathbf{F} - \mathbf{G}\,\mathbf{K})$ is singular for all $\mathbf{K} \in \mathcal{K}$ since the columns of the matrix in (7) provide a linearly dependent subset of the columns of $(\lambda_0\,\mathbf{I} - \mathbf{F} - \mathbf{G}\,\mathbf{K})$, and hence the eigenvalue λ_0 is fixed.

In order to prove the necessity, we shall employ the following notation and obvious notes. Let $\mathbf{H}_j$, $\mathbf{G}_j$, $j=1,2,\dots,q$, be given $n\times n_j$ and $n\times m_j$ matrices, and let $\mathcal{K}_j$ = class of all $m_j \times n_j$ real matrices.

NOTE 1. If the set of vectors $\{\mathbf{h}_1,\dots,\mathbf{h}_{j-1}\}$ and $\mathbf{g}$ are linearly independent and for some $\mathbf{h}_j \neq 0$, $\sum_{i=1}^{j}\alpha_i\,\mathbf{h}_i = \mathbf{0}$, then for rank k such that $\alpha_k \neq 0$, the set of vectors $\{\mathbf{h}_1,\dots,\mathbf{h}_k+\mathbf{g},\dots,\mathbf{h}_j\}$ is linearly independent.

The negation of (6) can be stated: For every non-empty subset $I = \{i_1,i_2,\dots,i_j\}$ of $\{1,2,\dots,m\}$

$$\text{rank}\ \left[\mathbf{H}_{i_1}(\lambda_0) \vdots \cdots \vdots \mathbf{H}_{i_j}(\lambda_0) \vdots \mathbf{G}_{i_1} \vdots \cdots \vdots \mathbf{G}_{i_j}\right] \geq \sum_{i=1}^{j} n_i, \tag{8}$$

for each $\lambda_0 \in \sigma(\mathbf{H})$.

NOTE 2. The condition (8) imply that for every non-empty subset $I = \{i_1, i_2,\dots, i_j\}$ of $\{1, 2 \dots, m\}$

$$\text{rank}\ \left[\mathbf{H}_{i_j}(\lambda_0) + \mathbf{G}_{i_j}\ \mathbf{K}_{i_j} \vdots \cdots \vdots \mathbf{H}_{i_j}(\lambda_0) + \mathbf{G}_{i_j}\ \mathbf{K}_{i_j} \vdots \mathbf{G}_{i_j} \vdots \cdots \vdots \mathbf{G}_{i_j}\right] \geq \sum_{i=1}^{j} n_i$$

for each $\lambda_0 \in \sigma(\mathbf{H})$ and any choice of the $\mathbf{K}_i$.

PROOF OF THE THEOREM. We assume (8) is truce, then we from a sequence $\mathbf{K}^j \in \mathcal{K}$ where at each step j, a $\Delta\mathbf{K}^j \in \mathcal{K}$ such that rank $[\mathbf{H} + \mathbf{G}\,\mathbf{K}^j] >$ rank $[\mathbf{H} + \mathbf{G}\,\mathbf{K}^{j-1}]$, where $\mathbf{K}^j = \mathbf{K}^{j-1} + \Delta\mathbf{K}^j$. By Note 2, the conditions (8) apply at each step. The proof is by induction on the number of steps j in this sequence. The case $j=1$ is trivial. Initially, choose $\mathbf{K}^1$ such that no column of $\mathbf{H} + \mathbf{G}\,\mathbf{K}^1$ is zero so that Note 1 can be applied at each step. Assume that the first $j-1$ columns of $\mathbf{H} + \mathbf{G}\,\mathbf{K}^{j-1} = \{\mathbf{h}_1,\dots,\mathbf{h}_n\}$ are linearly independent and column j is dependent, that is $\sum_{i=1}^{j}\alpha_i\mathbf{h}_i = \mathbf{0}$, and not all of the α_i are zero.

Let $\{\mathbf{h}_i\}_C$ denote the subset of $\{\mathbf{h}_1,\dots,\mathbf{h}_n\}$ associated with non-zero coefficients in the summation and $\{\mathbf{h}_i\}_{\underset{C}{\perp}}$ the remaining vectors. Let $\{\mathbf{g}\}_C$ be the subset of columns of $\mathbf{G}$ associated with $\{\mathbf{h}_i\}_C$ by the structural constraints $\mathcal{K}$, and $\{\mathbf{g}\}_{\underset{C}{\perp}}$ those associated with $\{\mathbf{h}_i\}_{\underset{C}{\perp}}$. From (8), we know there exists $\mathbf{g}_0 \neq 0$ such that $\mathbf{g}_0 \in \{\mathbf{g}\}_C + \{\mathbf{g}\}_{\underset{C}{\perp}}$ and $\mathbf{g}_0$ is linearly independent of $\{\mathbf{h}_1,\dots,\mathbf{h}_j\}$. If $\mathbf{g}_0 \in \{\mathbf{g}\}_C$, then we can invoke Note 1 to construct $\Delta\mathbf{K}^j$ so that the set $\{\mathbf{h}_1,\dots,\mathbf{h}_k + \gamma\,\mathbf{g}_0,\dots,\mathbf{h}_j\}$ is linearly independent, where we have chosen the index k without loss of generality.

On the other hand, suppose $\mathbf{g}_0 \in \{\mathbf{g}\}_{\underset{C}{\perp}}$, and no element of $\{\mathbf{g}\}_C$ is linearly independent of $\{\mathbf{h}_1, \dots, \mathbf{h}_n\}$. Since the set $\{\mathbf{h}_i\}_C$ is linearly dependent (8) guarantees the existence of $\mathbf{g}_0' \neq 0$, $\mathbf{g}_0' \in \{\mathbf{g}\}_C$, such that $\mathbf{g}_0'$ is linearly independent of $\{\mathbf{h}_i\}_C$. In this case, form $\Delta\mathbf{K}^j$ to add $\gamma\, \mathbf{g}_0'$, without loss of generality, to column ℓ to form the set $\{\mathbf{h}_i'\} = \{\mathbf{h}_1, \dots, \mathbf{h}_\ell + \gamma\, \mathbf{g}_0', \dots, \mathbf{h}_j\}$. Since $\mathbf{g}_0' \in \{\mathbf{g}\}_C$, for some set $\{\beta_i\}$ we must have $\mathbf{g}_0' = \sum_{i=1}^{j} \beta_i\, \mathbf{h}_j'$, where at least one of the β_i associated with $\{\mathbf{h}_i'\}_{\underset{C}{\perp}}$ must be non-zero. We have

$$0 = \sum_{i=1}^{j} \alpha_i\, \mathbf{h}_i = \sum_{i=1}^{j} \alpha_i\, \mathbf{h}_i' - \gamma\, \alpha_\ell\, \mathbf{g}_0' = \sum_{i=1}^{j} \alpha_i\, \mathbf{h}_i' - \gamma\, \alpha_\ell \sum_{i=1}^{t} \beta_i\, \mathbf{h}_i' = \sum_{i=1}^{j} (\alpha_i - \gamma\, \alpha_\ell\, \beta_i)\, \mathbf{h}_i'.$$

Obviously, we can choose γ in such a way that non-zero coefficients are associated with each of the $\{\mathbf{h}_i'\}_C$ and at least one of the $\{\mathbf{h}_i'\}_{\underset{C}{\perp}}$.

We can repeat this procedure, extending the number of non-zero coefficients in the summation until a non-zero coefficient is found associated with $\mathbf{g}_0$, at which point a $\Delta\mathbf{K}^P$ can be constructed which makes the set $\{\mathbf{h}_1', \dots, \mathbf{h}_j'\}$ linearly independent. The proof is complete.

References

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